J. Fluid Mech. (1967), *vol.* 28, *part* 2, *pp.* 391–411
Printed in Great Britain

Effect of finite boundaries on the Stokes resistance of an arbitrary particle

Part 3. Translation and rotation

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(Received 24 August 1966)

A general theory is given for the effect of solid walls on a translating *and rotating* particle in the limiting case of zero Reynolds number. Both the force *and couple* on the body are found as an expansion in terms of a parameter $\kappa = a/d$, assumed small, where a is a characteristic particle size and d a characteristic distance of the particle from walls. It is shown how such expansions may be used in specific examples. The theory is then extended to include the general motion of a small particle in an arbitrary Stokes flow field in which solid or other boundaries are present. Finally the motion of two small bodies of arbitrary shape in an arbitrary Stokes flow field is considered.

1. Introduction

Many authors have considered theoretically the effect of walls on the Stokesian motion of small solid particles when the particle and the walls have particularly simple shapes. Thus Cunningham (1910), Oseen (1927), Haberman & Sayre (1958) and Famularo (1961) have considered the motion of a sphere in the neighbourhood of solid walls whilst Wakiya (1957, 1959) has considered various wall-effect problems for an ellipsoidal particle. Chang (1961) seems to be the first author to consider bodies of more general shape. He derived a relation giving the drag on any body of revolution falling parallel to its symmetry axis at the centre of a circular cylinder. Brenner (1962, 1964*a*) considered the Stokes resistance of bodies of completely arbitrary shape moving in the neighbourhood of walls, the walls also being arbitrary. However he only considered the problem of finding *the force* on bodies which were undergoing translation *without rotation.*

The object of the present work is to obtain results for the wall effect for the force *and couple* acting upon a body of arbitrary shape, such a body possessing both translation *and rotation.* We shall express such results as an expansion in terms of a parameter κ, defined by the relation

$$\kappa = a/d, \tag{1.1}$$

where a is a characteristic particle dimension and d is a characteristic distance of the particle from the walls.

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Whereas the authors cited above all obtained expansions in this parameter κ by the use of the method of reflexions, we shall use an alternative method by forming inner and outer expansions. Although such a method is essentially equivalent to the method of reflexions, it will be found to be more convenient for our present purpose.

We shall first consider the flow field resulting from a body of arbitrary shape being placed in an arbitrary Stokes velocity field, the asymptotic value of the disturbance velocity field being found for large distances from the body. These results will then be used to obtain an expansion in κ for the force and couple acting on a body of arbitrary shape which is undergoing translation and rotation in a fluid at rest bounded by solid walls. Such an expansion will be found for *all* orders in κ.

It will be shown how such an expansion may be simplified by making use of body symmetry and wall symmetry. The uses and applications of such expansions in κ will be explained by the use of specific examples of wall effects.

The general theory is then extended to include the general motion of a particle of arbitrary shape placed in a fluid which is bounded by solid walls and which is also undergoing some arbitrary Stokes motion. As an example we consider an ellipsoid placed in a fluid between two moving parallel plates, the results obtained being compared with those obtained by Wakiya (1957) in his examination of this problem.

Finally, we consider two small solid bodies of arbitrary shape placed in an arbitrary Stokes flow field, no other solid walls being present. We obtain the forces and couples on these bodies again as an expansion in terms of a parameter κ defined by (1.1), where the length d is now defined to be the distance between the two particles.

2. Particles in arbitrary fields of flow

Into an undisturbed arbitrary Stokes velocity field $\bar{\mathbf{V}}'$, we suppose a solid body of arbitrary shape is placed. This body with surface B is assumed to move with a velocity $\mathbf{V}'$ and angular velocity $\mathbf{\Omega}'$. The dimensional velocity and pressure then occurring in the fluid are denoted by $\mathbf{v}'$ and p' respectively. In terms of a characteristic body dimension, a, and a characteristic fluid velocity U (which may be taken to be either $|\mathbf{V}'|$, $|a\mathbf{\Omega}'|$ or max. $|\bar{\mathbf{V}}'|$) dimensionless quantities $\bar{\mathbf{V}}$, $\mathbf{V}$, $\mathbf{\Omega}$, $\mathbf{v}$ and p may be defined corresponding respectively to $\bar{\mathbf{V}}'$, $\mathbf{V}'$, $\mathbf{\Omega}'$, $\mathbf{v}'$ and p' by the relations

$$\bar{\mathbf{V}} = \bar{\mathbf{V}}'/U; \quad \mathbf{V} = \mathbf{V}'/U; \quad \mathbf{\Omega} = \mathbf{\Omega}'a/U;$$
$$\mathbf{v} = \mathbf{v}'/U; \quad p = p'a/\mu U; \quad \mathbf{r} = \mathbf{r}'/a, \tag{2.1}$$

$\mathbf{r}'$ being the dimensional position vector. Thus $\mathbf{v}$ and p satisfy

$$\nabla^2\mathbf{v} - \nabla p = \mathbf{0}, \quad \nabla . \mathbf{v} = 0 \tag{2.2}$$

with the boundary conditions

$$\mathbf{v} \sim \bar{\mathbf{V}} \quad \text{as} \quad \mathbf{r} \to \infty; \quad \mathbf{v} = \mathbf{V} + \mathbf{\Omega} \wedge \mathbf{r} \quad \text{on} \quad B. \tag{2.3}$$

We define $\mathbf{u}$ to be the dimensionless disturbance velocity field, i.e.

$$\mathbf{u} = \mathbf{v} - \bar{\mathbf{V}}. \tag{2.4}$$

Thus, since $\overline{\mathbf{V}}$ must satisfy Stokes equations, it follows that $\mathbf{u}$ satisfies

$$\nabla^2\mathbf{u}-\nabla q = \mathbf{0}; \quad \nabla.\mathbf{u} = 0;$$
$$\mathbf{u}\to\mathbf{0} \quad \text{as} \quad \mathbf{r}\to\infty; \quad \mathbf{u} = \mathbf{V}+\mathbf{\Omega}\wedge\mathbf{r}-\overline{\mathbf{V}} \quad \text{on } B. \tag{2.5}$$

We suppose now that the solution $(\mathbf{u}, q)$ of (2.5) is known and that the corresponding stress tensor q_{ij}, say, has been calculated. We take $\overline{P}_{ij}$ to be the stress tensor corresponding to the undisturbed flow $\overline{\mathbf{V}}$.

Consider now the flow field $(\mathbf{u}^*, q^*)$ satisfying

$$\nabla^2\mathbf{u}^*-\nabla q^*+\boldsymbol{\lambda} = \mathbf{0}; \quad \nabla.\mathbf{u}^* = 0;$$
$$\mathbf{u}^*\to\mathbf{0} \quad \text{as} \quad \mathbf{r}\to\infty, \tag{2.6}$$

where these equations are valid *everywhere* and where $\boldsymbol{\lambda}$ is zero everywhere except on the surface B where it represents a surface distribution of force of magnitude $-(q_{ij}+\overline{P}_{ij})n_j$, where n_j is the *outward* unit normal to the surface B. The solution of the equations (2.6) must be unique. By direct substitution, it is seen that

$$\mathbf{u}^* = \mathbf{V}+\mathbf{\Omega}\wedge\mathbf{r}-\overline{\mathbf{V}}(\mathbf{r}) \quad \text{inside } B$$
$$\mathbf{u}^* = \mathbf{u} \qquad \text{outside } B \tag{2.7}$$

is a solution (2.6) and hence the only solution. Thus we may consider $(\mathbf{u}, q)$ to be the solution of equation (2.6) outside the surface B. The value of this velocity and pressure field may be expressed in terms of $\boldsymbol{\lambda}$ as a surface integral over B. Hence

$$\mathbf{u}(\mathbf{r}) = \frac{1}{8\pi}\int_B \left\{\frac{\boldsymbol{\lambda}}{|\mathbf{r}-\mathbf{r}'|}+\frac{\boldsymbol{\lambda}.(\mathbf{r}-\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right\}dS'$$
$$q(\mathbf{r}) = \frac{1}{4\pi}\int_B \frac{\boldsymbol{\lambda}.(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dS', \tag{2.8}$$

where $\mathbf{r}'$ is a general point on the surface B and dS' an element of surface area of B.

We now find the asymptotic expansion of $\mathbf{u}(\mathbf{r})$ and $q(\mathbf{r})$ for large $\mathbf{r}$ by noting that the function

$$f_{ij}(\mathbf{r},\mathbf{r}') = \left\{\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{r}'|}+\frac{(r_i-r_i')(r_j-r_j')}{|\mathbf{r}-\mathbf{r}'|^3}\right\} \tag{2.9}$$

may be written in the Taylor series

$$f_{ij}(\mathbf{r},\mathbf{r}') = f_{ij}(\mathbf{r},\mathbf{0})+r_k'[\partial f_{ij}/\partial r_k']_{\mathbf{r}'=\mathbf{0}}+\frac{1}{2!}r_k'r_l'[\partial^2 f_{ij}/\partial r_k'\partial r_l']_{\mathbf{r}'=\mathbf{0}}+\dots \tag{2.10}$$

if $|\mathbf{r}'| \ll |\mathbf{r}|$. From equation (2.9) we have relations such as

$$(\partial f_{ij}/\partial r_k')_{\mathbf{r}'=\mathbf{0}} = -\partial f_{ij}(\mathbf{r},\mathbf{0})/\partial r_k. \tag{2.11}$$

These relations may be used to reduce the equation (2.10) to the form

$$f_{ij}(\mathbf{r},\mathbf{r}') = s_{ij}-r_k'(\partial s_{ij}/\partial r_k)+\frac{1}{2!}r_k'r_l'[\partial^2 s_{ij}/(\partial r_k\,\partial r_l)]+\dots, \tag{2.12}$$

where
$$s_{ij} = r^{-1}(\delta_{ij}+r_ir_j/r^2). \tag{2.13}$$

The substitution of f_{ij} from equation (2.12) into the equation (2.8) for $\mathbf{u}(\mathbf{r})$ yields

$$u_j(\mathbf{r}) = \left\{\frac{1}{8\pi}\int_B \lambda_i dS'\right\}s_{ij}+\left\{-\frac{1}{8\pi}\int_B r_k'\lambda_i dS'\right\}\frac{\partial s_{ij}}{\partial r_k}+\dots. \tag{2.14}$$

In a similar manner $q(\mathbf{r})$ may be shown to possess the asymptotic expansion

$$q(\mathbf{r}) = \left\{\frac{1}{8\pi}\int_B \lambda_i dS'\right\} t_i + \left\{-\frac{1}{8\pi}\int_B r'_k \lambda_i dS'\right\} \frac{\partial t_i}{\partial r_k} + \ldots, \tag{2.15}$$

where

$$t_i = 2r_i/r^3. \tag{2.16}$$

Thus the asymptotic expansion of the disturbance field $(\mathbf{u}, q)$ for large r is

$$\begin{aligned} u_j &= A_i s_{ij} + A_{ik}(\partial s_{ij}/\partial r_k) + A_{ikl}[\partial^2 s_{ij}/(\partial r_k \partial r_l)] + \ldots, \\ q &= A_i t_i + A_{ik}(\partial t_i/\partial r_k) + A_{ikl}[\partial^2 t_i/(\partial r_k \partial r_l)] + \ldots, \end{aligned} \tag{2.17}$$

where the tensors $A_i, A_{ik}, A_{ikl}, \ldots$, are dependent upon the shape of the body B as well as being dependent upon the quantities $\mathbf{V}$, $\mathbf{\Omega}$ and $\overline{\mathbf{V}}(\mathbf{r})$ in a linear manner.

3. Wall effects

In this section we consider the Stokesian motion of a small solid particle in a stationary fluid in which there are solid walls present. We let a be a characteristic dimension of the body B and d be a characteristic distance of the body from the walls W. The ratio $\kappa = a/d$ is assumed to be very small. All quantities are measured relative to a set of fixed axes at an origin O within the body B. We suppose the velocity and angular velocity of the body (at O) are respectively $\mathbf{V}'$ and $\mathbf{\Omega}'$. The velocity and pressure occurring at a general point in the fluid are taken to be $\mathbf{v}'$ and p'. Dimensionless (undashed) quantities are now defined in a manner similar to that of § 2 by the length a, a characteristic velocity U (equal to either $|\mathbf{V}'|$ or $|a\mathbf{\Omega}'|$) and the viscosity μ.

The dimensionless velocity $\mathbf{v}$ and pressure p of the fluid then satisfy

$$\nabla^2\mathbf{v} - \nabla p = \mathbf{0}, \quad \nabla . \mathbf{v} = 0, \tag{3.1}$$

with the boundary conditions

$$\left.\begin{aligned} &\mathbf{v} = \mathbf{0} \quad \text{on } W; \quad \mathbf{v} \to \mathbf{0} \quad \text{as} \quad \mathbf{r} \to \infty; \\ &\mathbf{v} = \mathbf{V} + \mathbf{\Omega} \wedge \mathbf{r} \quad \text{on } B, \end{aligned}\right\} \tag{3.2}$$

where the walls W are at a distance κ^{-1} from the body.

In order to solve this problem, we define inner and outer expansions. The inner expansion

$$\left.\begin{aligned} \mathbf{v} &= \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \ldots, \\ p &= p_0 + p_1 + p_2 + \ldots, \end{aligned}\right\} \tag{3.3}$$

is defined to be the solution of the equations

$$\left.\begin{aligned} &\nabla^2\mathbf{v} - \nabla p = \mathbf{0}, \quad \nabla . \mathbf{v} = 0, \\ &\mathbf{v} = \mathbf{V} + \mathbf{\Omega} \wedge \mathbf{r} \quad \text{on } B, \end{aligned}\right\} \tag{3.4}$$

the outer boundary condition being obtained from the required matching onto the outer expansion. In order to define this outer expansion, we first define a new independent variable $\tilde{\mathbf{r}}$ by the relation

$$\tilde{\mathbf{r}} = \kappa\mathbf{r}. \tag{3.5}$$

The outer expansion

$$\mathbf{v} = \tilde{\mathbf{v}}_1 + \tilde{\mathbf{v}}_2 + \ldots, \quad p = \tilde{p}_1 + \tilde{p}_2 + \ldots, \tag{3.6}$$

is made to satisfy

$$\left.\begin{aligned} \tilde{\nabla}^2\mathbf{v} - \tilde{\nabla}(p/\kappa) = \mathbf{0}, \quad \tilde{\nabla}\,.\,\mathbf{v} = 0, \\ \mathbf{v} = \mathbf{0} \quad \text{on } W, \quad \mathbf{v} \to \mathbf{0} \quad \text{as} \quad \mathbf{r} \to \infty. \end{aligned}\right\} \tag{3.7}$$

We let $\mathbf{F}_0$, $\mathbf{F}_1$, $\mathbf{F}_2, \ldots$, be the dimensionless forces (equal to $\mathbf{F}'_0/\mu a U$, $\mathbf{F}'_1/\mu a U \ldots$) acting on the body due to the flow fields $(\mathbf{v}_0, p_0)$, $(\mathbf{v}_1, p_1), \ldots$, etc. Similarly $\mathbf{G}_0$, $\mathbf{G}_1, \mathbf{G}_2, \ldots$, are defined to be the dimensionless couples (equal to $\mathbf{G}'_0/\mu a^2 U$, $\mathbf{G}'_1/\mu a^2 U, \ldots$) acting on the body due to $(\mathbf{v}_0, p_0)$, $(\mathbf{v}_1, p_1)$, $(\mathbf{v}_2, p_2), \ldots$, respectively. We write $\mathbf{F}$ and $\mathbf{G}$ to be the total dimensionless force and couple acting on the body. The six-dimensional force-couple vectors $\mathfrak{F}$, $\mathfrak{F}_0$, $\mathfrak{F}_1, \ldots$, are defined to be

$$\mathfrak{F} = \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix}; \quad \mathfrak{F}_0 = \begin{pmatrix} \mathbf{F}_0 \\ \mathbf{G}_0 \end{pmatrix}; \quad \mathfrak{F}_1 = \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{G}_1 \end{pmatrix}, \; \ldots. \tag{3.8}$$

Then, by definition

$$\mathfrak{F} = \mathfrak{F}_0 + \mathfrak{F}_1 + \mathfrak{F}_2 + \ldots. \tag{3.9}$$

The six dimensional velocity-angular velocity vector of the body $\mathfrak{V}$ is defined by the relation

$$\mathfrak{V} = \begin{pmatrix} \mathbf{V} \\ \mathbf{\Omega} \end{pmatrix}. \tag{3.10}$$

We define uniquely the first term $(\mathbf{v}_0, p_0)$ of the inner expansion by requiring that $\mathbf{v}_0 \to \mathbf{0}$ as $\mathbf{r} \to \infty$. Thus

$$\left.\begin{aligned} \nabla^2\mathbf{v}_0 - \nabla p_0 = \mathbf{0}, \quad \nabla\,.\,\mathbf{v}_0 = 0, \\ \mathbf{v}_0 = \mathbf{V} + \mathbf{\Omega} \wedge \mathbf{r} \quad \text{on } B, \quad \mathbf{v}_0 \to \mathbf{0} \quad \text{as} \quad \mathbf{r} \to \infty. \end{aligned}\right\} \tag{3.11}$$

By equations (2.17), it is seen that for large $\mathbf{r}$, $(\mathbf{v}_0, p_0)$ is of the form

$$\left.\begin{aligned} (v_0)_j &= A_i s_{ij} + A_{ik}(\partial s_{ij}/\partial r_k) + \ldots \\ p_0 &= A_i t_i + A_{ik}(\partial t_i/\partial r_k) + \ldots, \end{aligned}\right\} \tag{3.12}$$

where the tensors A_i, $A_{ik}, \ldots$, are dependent on the body shape as well as being linearly dependent on the vectors $\mathbf{V}$ and $\mathbf{\Omega}$ and hence upon the six-dimensional vector $\mathfrak{V}$. Thus there exist quantities $(l_1)_{il}$, $(l_2)_{ikl}, \ldots$, etc., dependent only upon body shape such that

$$A_i = (l_1)_{il}\mathfrak{V}_l; \quad A_{ik} = (l_2)_{ikl}\mathfrak{V}_l, \ldots, \tag{3.13}$$

where the index l takes the values 1–6. For $l = 1, 2, 3$ the quantities $(l_1)_{il}$, $(l_2)_{ikl}, \ldots$, etc., are true tensors whereas for $l = 4, 5, 6$ they are pseudotensors.

Thus for large $\mathbf{r}$, the asymptotic form of $(\mathbf{v}_0, p_0)$ is

$$\left.\begin{aligned} (v_0)_j &= (l_1)_{il}\mathfrak{V}_l s_{ij} + (l_2)_{ikl}\mathfrak{V}_l(\partial s_{ij}/\partial r_k) + \ldots, \\ p_0 &= (l_1)_{il}\mathfrak{V}_l t_i + (l_2)_{ikl}\mathfrak{V}_l(\partial t_i/\partial r_k) + \ldots, \end{aligned}\right\} \tag{3.14}$$

which when expressed in outer variables, takes the form

$$\left.\begin{aligned} (v_0)_j &= \kappa(l_1)_{il}\mathfrak{V}_l \tilde{s}_{ij} + \kappa^2(l_2)_{ikl}\mathfrak{V}_l(\partial \tilde{s}_{ij}/\partial \tilde{r}_k) + \ldots, \\ (p_0/\kappa) &= \kappa(l_1)_{il}\mathfrak{V}_l \tilde{t}_i + \kappa^2(l_2)_{ikl}\mathfrak{V}_l(\partial \tilde{t}_i/\partial \tilde{r}_k) + \ldots, \end{aligned}\right\} \tag{3.15}$$

where $\tilde{s}_{ij}$ and $\tilde{t}_i$ are the same expressions as s_{ij} and t_i with $\mathbf{r}$ replaced by $\tilde{\mathbf{r}}$.

The first term in the outer expansion ($\tilde{\mathbf{v}}_1$, p_1) satisfies the equations (3.7) with $\tilde{\mathbf{v}}_1$ and $\tilde{p}_1$ for small values of $\tilde{\mathbf{r}}$ having the form (3.15). Writing

$$\left.\begin{aligned}(\tilde{v}_1^*)_j &= (\tilde{v}_1)_j - \kappa(l_1)_{il}\mathfrak{B}_l\tilde{s}_{ij} - \kappa^2(l_2)_{ikl}\mathfrak{B}_l(\partial\tilde{s}_{ij}/\partial\tilde{r}_k) - \ldots,\\ (\tilde{p}_1^*/\kappa) &= (\tilde{p}_1/\kappa) - \kappa(l_1)_{il}\mathfrak{B}_l\tilde{t}_i - \kappa^2(l_2)_{ikl}\mathfrak{B}_l(\partial\tilde{t}_i/\partial\tilde{r}_k) - \ldots,\end{aligned}\right\} \tag{3.16}$$

we see that ($\tilde{\mathbf{v}}_1^*$, $\tilde{p}_1^*$) satisfies Stokes equations with the boundary conditions

$$\left.\begin{aligned}&(\tilde{v}_1^*)_j \quad \text{bounded as} \quad \tilde{\mathbf{r}}\to\mathbf{0},\\ &(\tilde{v}_1^*)_j \sim -\kappa(l_1)_{il}\mathfrak{B}_l\tilde{s}_{ij} - \kappa^2(l_2)_{ikl}\mathfrak{B}_l(\partial\tilde{s}_{ij}/\partial\tilde{r}_k) - \ldots \quad \text{as} \quad \tilde{\mathbf{r}}\to\infty,\\ \text{and}\quad &(\tilde{v}_1^*)_j = -\kappa(l_1)_{il}\mathfrak{B}_l\tilde{s}_{ij} - \kappa^2(l_2)_{ikl}\mathfrak{B}_l(\partial\tilde{s}_{ij}/\partial\tilde{r}_k) - \ldots \quad \text{on } W.\end{aligned}\right\} \tag{3.17}$$

Thus expanding $(\tilde{v}_1^*)_p$ in a Taylor series about $\tilde{\mathbf{r}} = \mathbf{0}$, we see that

$$(\tilde{v}_1^*)_p = B_p + \tilde{r}_q B_{qp} + \tilde{r}_s\tilde{r}_q B_{qsp} + \ldots \quad \text{as} \quad \tilde{\mathbf{r}}\to\mathbf{0}, \tag{3.18}$$

where $B_p, B_{qp}, B_{qsp}, \ldots$, are linear functions of the quantities $\kappa(l_1)_{il}\mathfrak{B}_l$, $\kappa^2(l_2)_{ikl}\mathfrak{B}_l \ldots$, etc., appearing in the boundary conditions (3.17). Thus we may write

$$\left.\begin{aligned}B_p &= ({}_1L_1)_{pj}\kappa(l_1)_{jl}\mathfrak{B}_l + ({}_1L_2)_{pjk}\kappa^2(l_2)_{kjl}\mathfrak{B}_l + \ldots,\\ B_{qp} &= ({}_2L_1)_{qpj}\kappa(l_1)_{jl}\mathfrak{B}_l + ({}_2L_2)_{qpjk}\kappa^2(l_2)_{kjl}\mathfrak{B}_l + \ldots,\\ &\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\end{aligned}\right\} \tag{3.19}$$

where $({}_1L_1)_{pj}$, $({}_1L_2)_{pjk}$, $({}_2L_1)_{qpj}$, ..., etc., are tensors which are functions only of the shape of the boundaries W.

Equation (3.18) when expressed in terms of inner variables, takes the form

$$(\tilde{v}_1^*)_p = B_p + \kappa r_q B_{qp} + \kappa^2 r_s r_q B_{qsp} + \ldots. \tag{3.20}$$

The next term in the inner expansion ($\mathbf{v}_1$, p_1) satisfies Stokes equations and is made to satisfy the boundary conditions

$$\left.\begin{aligned}&(v_1)_p = 0 \quad \text{on } B,\\ &(v_1)_p \sim B_p + \kappa r_q B_{qp} + \kappa^2 r_s r_q B_{qsp} + \ldots \quad \text{as} \quad \mathbf{r}\to\infty.\end{aligned}\right\} \tag{3.21}$$

Writing $(v_1^*)_p = (v_1)_p - B_p - \kappa r_q B_{qp} - \ldots$, it may be seen that by making use of the results of §2, the asymptotic form of $\mathbf{v}_1^*$ for large $\mathbf{r}$ is

$$(v_1^*)_j = C_i s_{ij} + C_{ik}(\partial s_{ij}/\partial r_k) + \ldots, \tag{3.22}$$

where

$$\left.\begin{aligned}C_i &= ({}_1q_1)_{ip}B_p + \kappa({}_1q_2)_{ipq}B_{qp} + \ldots,\\ C_{ik} &= ({}_2q_1)_{ikp}B_p + \kappa({}_2q_2)_{ikpq}B_{qp} + \ldots,\\ &\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\end{aligned}\right\} \tag{3.23}$$

the tensors $({}_1q_1)_{ip}$, $({}_1q_2)_{ipq}$, ..., being functions only of the shape of the body B. It should be noted that these tensors $({}_1q_1)_{ip}$, $({}_1q_2)_{ipq}$ need not necessarily be unique because there exist relations between the coefficients B_{qp}, B_{qsp} in (3.18) since $\tilde{\mathbf{v}}_1^*$ itself must satisfy Stokes equations (i.e. we have $\nabla . \tilde{\mathbf{v}}_1^* = 0$ and $\nabla \wedge \nabla^2 \mathbf{v}_1 = \mathbf{0}$). However, this non-uniqueness need not concern us since any choice of values for these tensors should result in the same flow field (3.22).

From the definition of the quantities ${}_n\mathbf{q}_1$ and ${}_n\mathbf{q}_2$ (see equation (3.23)) it is seen that in polyadic notation

$$\mathbf{l}_n = (-{}_n\mathbf{q}_1, \quad -{}_n\mathbf{q}_2 : \boldsymbol{\epsilon}). \tag{3.24}$$

$(\mathbf{v}_0, p_0)$ is given by the equations (3.14). Thus the force $\mathbf{F}_0$ and couple $\mathbf{G}_0$ due to this flow field is

$$(F_0)_j = -8\pi(l_1)_{jl}\mathfrak{B}_l, \quad (G_0)_j = -8\pi\epsilon_{jik}(l_2)_{ikl}\mathfrak{B}_l; \tag{3.25}$$

$\mathbf{v}_1$ is given by

$$(v_1)_p = \{B_p + \kappa r_q B_{qp} + \ldots\} + C_i s_{ij} + C_{ik}(\partial s_{ij}/\partial r_k) + \ldots,$$

where C_i, C_{ik}, ..., are given by the equations (3.19) and (3.23). Thus $\mathbf{F}_1$ and $\mathbf{G}_1$ are given by

$$\left.\begin{aligned}(F_1)_j &= -8\pi\{({}_1q_1)_{jp}B_p + \kappa({}_1q_2)_{jpq}B_{qp} + \ldots\},\\ (G_1)_j &= -8\pi\epsilon_{jik}\{({}_2q_1)_{ikp}B_p + \kappa({}_2q_2)_{ikpq}B_{qp} + \ldots\}.\end{aligned}\right\} \tag{3.26}$$

Hence the six-dimensional force-couple tensors $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are given (in polyadic notation) by the relations

$$\mathfrak{F}_0 = -\mathfrak{R}.\mathfrak{B}, \tag{3.27}$$

$$\mathfrak{F}_1 = \sum_{m,n} \mathbf{p}_{n+1}[n+1]\,({}_{n+1}\mathbf{L}_m)\,[m]\,\mathbf{l}_m.\mathfrak{B}\kappa^{m+n}, \tag{3.28}$$

where the summation is taken over all integers m, n such that $m \geqslant 1$, $n \geqslant 0$. $\mathfrak{R}$ is the six-dimensional resistance tensor defined as

$$\mathfrak{R} = 8\pi\begin{pmatrix}+\mathbf{l}_1\\ -\boldsymbol{\epsilon}:\mathbf{l}_2\end{pmatrix},$$

or in terms of the tensors ${}_1\mathbf{q}_1$, ${}_1\mathbf{q}_2$, ..., etc. as

$$\mathfrak{R} = 8\pi\begin{pmatrix}-{}_1\mathbf{q}_1 & -{}_1\mathbf{q}_2:\boldsymbol{\epsilon}\\ +\boldsymbol{\epsilon}:{}_2\mathbf{q}_1 & +\boldsymbol{\epsilon}:{}_2\mathbf{q}_2:\boldsymbol{\epsilon}\end{pmatrix}. \tag{3.29}$$

The quantity $\mathbf{p}_{n+1}$ appearing in equation (3.28) consists of two $(n+2)$ order tensors, one being a true tensor and the other a pseudotensor. Thus $\mathbf{p}_{n+1}$ is defined as

$$\mathbf{p}_{n+1} = 8\pi\begin{pmatrix}-{}_1\mathbf{q}_{n+1}\\ +\boldsymbol{\epsilon}:{}_2\mathbf{q}_{n+1}\end{pmatrix}, \tag{3.30}$$

i.e. it is a quantity defined by $(n+2)$ suffixes all but the first running from 1 to 3, the first suffix itself running from 1 to 6. For this first suffix running from 1 to 3, $\mathbf{p}_{n+1}$ is the true tensor $(8\pi\,{}_1\mathbf{q}_{n+1})$, whilst for the first suffix running from 4 to 6 it is the pseudotensor $(-8\pi\boldsymbol{\epsilon}:{}_2\mathbf{q}_{n+1})$.

In a similar manner we may repeat the above arguments to find $(\mathbf{v}_2, p_2)$ and hence $\mathfrak{F}_2$. Thus we obtain

$$\mathfrak{F}_2 = \sum_{abcd} \mathbf{p}_{a+1}[a+1]({}_{a+1}\mathbf{L}_b)\,[b]\,({}_b\mathbf{q}_{c+1})\,[c+1]\,({}_{c+1}\mathbf{L}_d)\,[d]\,\mathbf{l}_d.\mathfrak{B}\kappa^{a+b+c+d}, \tag{3.31}$$

the summation being over all integers a, b, c, d such that $a, c \geqslant 0$; $b, d \geqslant 1$.

By repeating this process, the complete expansion for the force-couple vector $\mathfrak{F}$ may be obtained as

$$\begin{aligned}\mathfrak{F} = -\mathfrak{R}.\mathfrak{B} + \Sigma\,\kappa^{a+b+\ldots+n}\,\mathbf{p}_{a+1}[a+1]\,({}_{a+1}\mathbf{L}_b)\,[b]\\ \times({}_b\mathbf{q}_{c+1})\,[c+1]\ldots[l]\,({}_l\mathbf{q}_{m+1})\,[m+1]\,({}_{m+1}\mathbf{L}_n)\,[n]\,\mathbf{l}_n.\mathfrak{B},\end{aligned} \tag{3.32}$$

the summation being taken over all integers a, b, ..., n, satisfying

$$a,\ c,\ \ldots,\ m \geqslant 0, \quad b,\ \ldots,\ l,\ n \geqslant 1. \tag{3.33}$$

It should be noted that the above arguments remain valid if some of the walls W, instead of being solid, are free surfaces. The equation (3.32) would therefore still apply in such cases.

In the following section, it will be demonstrated how the expansion (3.32) may be used in specific examples.

Note on polyadic notation

If $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are the unit base vectors of the orthogonal co-ordinate system used, then a polyadic $\mathbf{P}$ may be written as

$$\mathbf{P} = \sum_{ab\ldots n} (\mathbf{i}_a \mathbf{i}_b \ldots \mathbf{i}_n) P_{ab\ldots n},$$

where the scalar quantities $P_{ab\ldots n}$ may be considered as the components of the tensor corresponding to the polyadic $\mathbf{P}$. The direct product $\mathbf{PQ}$ of the polyadic $\mathbf{P}$ with another polyadic $\mathbf{Q}$ corresponding to a tensor $Q_{pq\ldots w}$, is defined by

$$\mathbf{PQ} = \sum_{ab\ldots n} \sum_{pq\ldots w} (\mathbf{i}_a \mathbf{i}_b \ldots \mathbf{i}_m \mathbf{i}_n)(\mathbf{i}_p \mathbf{i}_q \ldots \mathbf{i}_w) P_{ab\ldots n} Q_{pq\ldots w}$$

the contracted products $\mathbf{P}.\mathbf{Q}$ and $\mathbf{P}:\mathbf{Q}$ being defined as

$$\mathbf{P}.\mathbf{Q} = \sum_{ab\ldots n} \sum_{pq\ldots w} (\mathbf{i}_a \mathbf{i}_b \ldots \mathbf{i}_l \mathbf{i}_m)(\mathbf{i}_n . \mathbf{i}_p)(\mathbf{i}_q \mathbf{i}_r \ldots \mathbf{i}_w) P_{ab\ldots mn} Q_{pq\ldots w}$$

$$= \sum_{ab\ldots mnq\ldots w} (\mathbf{i}_a \mathbf{i}_b \ldots \mathbf{i}_l \mathbf{i}_m)(\mathbf{i}_q \mathbf{i}_r \ldots \mathbf{i}_w) P_{ab\ldots mn} Q_{nqr\ldots w}$$

and

$$\mathbf{P}:\mathbf{Q} = \sum_{ab\ldots lmnr\ldots w} (\mathbf{i}_a \mathbf{i}_b \ldots \mathbf{i}_l)(\mathbf{i}_r \ldots \mathbf{i}_w) P_{ab\ldots lmn} Q_{nmr\ldots w}$$

with similar definitions for $\mathbf{P}[3]\mathbf{Q}$, $\mathbf{P}[4]\mathbf{Q}\ldots$, there being n-contractions in a product $\mathbf{P}[n]\mathbf{Q}$.

The above notation is that used by Milne (1957) and Chapman & Cowling (1961).

4. Examples of wall effects

The force $\mathbf{F}$ and couple $\mathbf{G}$ acting upon a non-rotating body for which $\mathbf{\Omega} = \mathbf{0}$, may by the equations (3.24), (3.30) and (3.32), be shown to be given by

$$\mathbf{F}/8\pi = +{}_1\mathbf{q}_1 . \mathbf{V} + \kappa[{}_1\mathbf{q}_1 . {}_1\mathbf{L}_1 . {}_1\mathbf{q}_1] . \mathbf{V} + \kappa^2[{}_1\mathbf{q}_1 . {}_1\mathbf{L}_2 : {}_2\mathbf{q}_1 + {}_1\mathbf{q}_2 : {}_2\mathbf{L}_1 . {}_1\mathbf{q}_1 + {}_1\mathbf{q}_1 . {}_1\mathbf{L}_1 . {}_1\mathbf{q}_1 . {}_1\mathbf{L}_1 . {}_1\mathbf{q}_1] . \mathbf{V} + O(\kappa^3), \quad (4.1)$$

$$\mathbf{G}/8\pi = -\boldsymbol{\epsilon} : {}_2\mathbf{q}_1 . \mathbf{V} - \kappa[\boldsymbol{\epsilon} : {}_2\mathbf{q}_1 . {}_1\mathbf{L}_1 . {}_1\mathbf{q}_1] . \mathbf{V} - \kappa^2\boldsymbol{\epsilon} : [{}_2\mathbf{q}_1 . {}_1\mathbf{L}_2 : {}_2\mathbf{q}_1 + {}_2\mathbf{q}_2 : {}_2\mathbf{L}_1 . {}_1\mathbf{q}_1 + {}_2\mathbf{q}_1 . {}_1\mathbf{L}_1 . {}_1\mathbf{q}_1 . {}_1\mathbf{L}_1 . {}_1\mathbf{q}_1] . \mathbf{V} + O(\kappa^3). \quad (4.2)$$

Similarly for a body undergoing a rotation without translation, the force $\mathbf{F}$ and couple $\mathbf{G}$ are given by

$$\mathbf{F}/8\pi = +{}_1\mathbf{q}_2 : \boldsymbol{\epsilon} . \mathbf{\Omega} + \kappa[{}_1\mathbf{q}_1 . {}_1\mathbf{L}_1 . {}_1\mathbf{q}_2 : \boldsymbol{\epsilon}] . \mathbf{\Omega} + \kappa^2[{}_1\mathbf{q}_1 . {}_1\mathbf{L}_2 : {}_2\mathbf{q}_2 + {}_1\mathbf{q}_2 : {}_2\mathbf{L}_1 . {}_1\mathbf{q}_2 + {}_1\mathbf{q}_1 . {}_1\mathbf{L}_1 . {}_1\mathbf{q}_1 . {}_1\mathbf{L}_1 . {}_1\mathbf{q}_2] : \boldsymbol{\epsilon} . \mathbf{\Omega} + O(\kappa^3), \quad (4.3)$$

$$\mathbf{G}/8\pi = -\boldsymbol{\epsilon} : {}_2\mathbf{q}_2 : \boldsymbol{\epsilon} . \mathbf{\Omega} - \kappa[\boldsymbol{\epsilon} : {}_2\mathbf{q}_1 . {}_1\mathbf{L}_1 . {}_1\mathbf{q}_2 : \boldsymbol{\epsilon}] . \mathbf{\Omega} - \kappa^2\boldsymbol{\epsilon} : [{}_2\mathbf{q}_1 . {}_1\mathbf{L}_2 : {}_2\mathbf{q}_2 + {}_2\mathbf{q}_2 : {}_2\mathbf{L}_1 . {}_1\mathbf{q}_2 + {}_2\mathbf{q}_1 . {}_1\mathbf{L}_1 . {}_1\mathbf{q}_1 . {}_1\mathbf{L}_1 . {}_1\mathbf{q}_2] : \boldsymbol{\epsilon} . \mathbf{\Omega} + O(\kappa^3). \quad (4.4)$$

These formulae for the force and couple acting on the body may be simplified should the body possess symmetry properties about the chosen origin. For example, if the body were centrally symmetric the tensor ${}_r\mathbf{q}_s$ would be identically zero if $(r+s)$ were odd. In a similar manner, if the walls W should possess symmetry about the origin, this would place a restriction upon the form taken by the tensors ${}_r\mathbf{L}_s$.

As a specific example we consider a body which is axially symmetric about an axis 1, which also possesses fore–aft symmetry about the origin. Such a body we assume to be translating without rotation in the direction of the axis 2, in a fluid

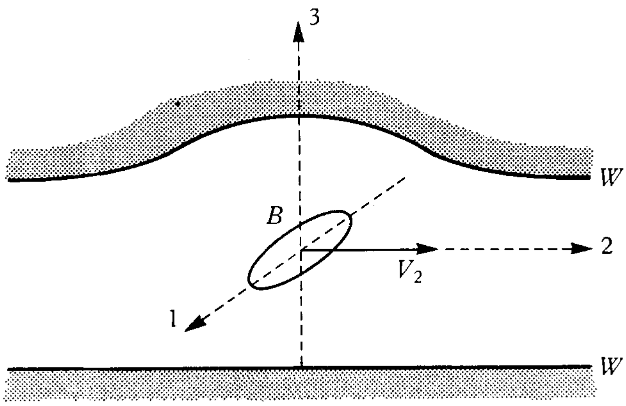

FIGURE 1

bounded by a system of walls W which are axisymmetric about the axis 3 (see figure 1). Then by the use of equation (3.32), we may find the force $\mathbf{F}$ and couple $\mathbf{G}$ on the body as

$$\mathbf{F} = (0, F_2, 0), \quad \mathbf{G} = (G_1, 0, 0), \tag{4.5}$$

where

$$F_2/8\pi = [1+\kappa\{({}_1q_1)_{22}\,({}_1L_1)_{22}\}+\kappa^2\{({}_1q_1)_{22}\,({}_1L_1)_{22}\}^2+\kappa^3\{({}_1q_1)_{22}\,({}_1L_1)_{22}\}^3]({}_1q_1)_{22}\,V_2$$
$$+\kappa^3[({}_1q_1)_{22}\,({}_1L_3)_{22ij}\,({}_3q_1)_{ji22}+({}_1q_3)_{22ij}\,({}_3L_1)_{ji22}\,({}_1q_1)_{22}]\,V_2+O(\kappa^4),$$
$$G_1/8\pi = -\kappa^2\{\epsilon_{1ij}\,({}_2q_2)_{ji23}\,({}_2L_1)_{322}\,({}_1q_1)_{22}+\epsilon_{1ij}\,({}_2q_2)_{ji32}\,({}_2L_1)_{232}\,({}_1q_1)_{22}\}$$
$$\times\{1+\kappa({}_1q_1)_{22}\,({}_1L_1)_{22}\}\,V_2+O(\kappa^4).$$

Hence F_2 and G_1 are of the form

$$F_2 = \frac{V_2}{a+b\kappa+c\kappa^3}+O(\kappa^4), \quad G_1 = \frac{V_2\,d\kappa^2}{a+b\kappa}+O(\kappa^4), \tag{4.6}$$

where a, b, c, d are constants. This result is in complete agreement with that obtained by Wakiya (1957) for the force and couple acting upon a spheroid translating in a fluid contained between two parallel planes.

As a second example, we consider the pure rotation of a sphere about an axis 1, in a fluid bounded by a system of walls which is axisymmetric about the axis 1 (see figure 2). We take axes 2 and 3 through the centre of the sphere. Relative to spherical polar axes at the sphere centre based on the axis 1, the only non-zero component of fluid velocity is v_ϕ and this satisfies the equation

$$r\frac{\partial^2}{\partial r^2}(rv_\phi)+\frac{\partial}{\partial\theta}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi)\right\} = 0.$$

The solution of this equation is

$$v_\phi = (Ar^n + Br^{-n-1})P_n(\cos\theta),$$

where n is an integer.

Thus we see that if a sphere is axially situated in an axisymmetric flow which increases like r^{+n} at infinity, then such a flow must be of the form

$$v_\phi = Ar^n P_n(\cos\theta)$$

and the disturbance flow due to the sphere must be

$$v_\phi = -Ar^{-n-1}P_n(\cos\theta).$$

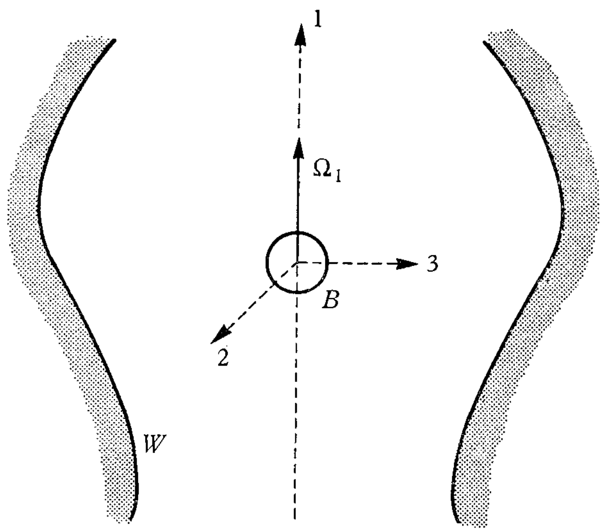

FIGURE 2

Thus for examples of the type which we are now considering, the tensor ${}_r\mathbf{q}_s$ must be identically zero for all r and s unless $r = s$. This result greatly simplifies the equation (3.32). Thus it may be shown that $\mathbf{G} = (G_1, 0, 0)$ where G_1 is given by

$$G_1/8\pi = [1+\kappa^3\{4({}_2\mathbf{q}_2 : {}_2\mathbf{L}_2)_{3232}\} + \kappa^6\{4({}_2\mathbf{q}_2 : {}_2\mathbf{L}_2)_{3232}\}^2 + \kappa^9\{4({}_2\mathbf{q}_2 : {}_2\mathbf{L}_2)_{3232}\}^3]$$
$$\times ({}_2q_2)_{23ij}\,\epsilon_{ji1}\,\Omega_1 - \kappa^8[\boldsymbol{\epsilon} : {}_2\mathbf{q}_2 : {}_2\mathbf{L}_3[3]\,{}_3\mathbf{q}_3[3]\,{}_3\mathbf{L}_2 : {}_2\mathbf{q}_2 : \epsilon]\,.\,\boldsymbol{\Omega} + O(\kappa^{10}).$$

Thus if ${}^\infty G_1$ is the value of G_1 which one would expect in the absence of walls (i.e. for $\kappa = 0$), then we see that G_1 is given in terms of ${}^\infty G_1$ by the relation

$$\frac{G_1}{{}^\infty G_1} = \frac{1}{1+a\kappa^3+b\kappa^8} + O(\kappa^{10}), \tag{4.7}$$

where a and b are constants. Should the walls W possess fore–aft symmetry about the plane 1 (containing axes 2 and 3), then G_1 may be shown to be of the form

$$\frac{G_1}{{}^\infty G_1} = \frac{1}{1+a\kappa^3+c\kappa^{10}+d\kappa^{14}} + O(\kappa^{17}), \tag{4.8}$$

where a, c and d are constants. The solution (4.7) is in agreement with the results obtained by Brenner (1964*c*) for a rotating sphere in the neighbourhood of a solid plane wall (or free surface). An example for which the walls W possessed fore–aft symmetry was also investigated by Brenner & Sonshine (1964). This concerned a sphere rotating at the centre of an infinitely long circular cylinder, for which it was shown that the couple acting on the sphere was exactly of the form given by the equation (4.8).

We now investigate a very specific example in which a small spherical particle rotates about an axis 1 in a fluid contained in a solid spherical vessel concentric with the particle (figure 3). As before the tensor ${}_r\mathbf{q}_s$ must be identically zero for

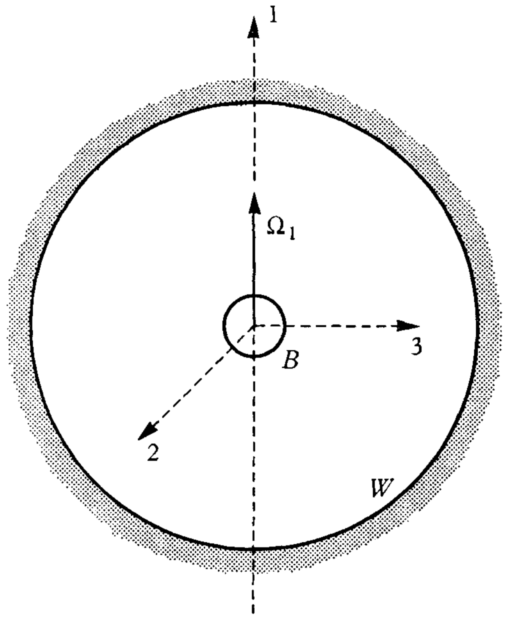

FIGURE 3

all r and s unless $r = s$. Since the walls W are now spherical with centre at the origin, it may be proved in a likewise manner that the tensor ${}_r\mathbf{L}_s$ is identically zero unless $r = s$. The equation (3.32) then gives the couple G_1 on the particle as the expansion

$$G_1 = +8\pi({}_2q_2)_{23ij}\,\epsilon_{ji1}\,\Omega_1 \sum_{n=0}^{\infty} \{4({}_2\mathbf{q}_2 : {}_2\mathbf{L}_2)_{3232}\,\kappa^3\}^n, \tag{4.9}$$

this relation being the *entire* expansion. However, the series occurring in this equation is a geometric series and may therefore be summed to give

$$G_1 = \frac{8\pi({}_2q_2)_{23ij}\,\epsilon_{ji1}\,\Omega_1}{1 - 4({}_2\mathbf{q}_2 : {}_2\mathbf{L}_2)_{3232}\,\kappa^3}. \tag{4.10}$$

This result should be compared with the *exact* solution to this problem obtained by Landau & Lifshitz (1959) without making the assumption that $\kappa \ll 1$. They obtained the expression

$$G_1 = \frac{-8\pi\Omega_1}{1-\kappa^3}, \tag{4.11}$$

which is exactly analogous to equation (4.10).

5. Wall effects with arbitrary fields of flow

In this section we extend the theory described in §3 to problems in which a small particle of arbitrary shape is placed in a fluid which is undergoing some arbitrary Stokesian motion, there also being a system of walls W present. We let the undisturbed fluid motion be $\mathbf{U}$ (in dimensionless variables), which we assume to possess a length scale of the same order of magnitude as the distance d from the particle to wall. Thus we shall take $\mathbf{U}$ to be a function of $\tilde{\mathbf{r}}$. Now $\mathbf{U}$ and the corresponding pressure P must satisfy Stokes equations together with the required boundary conditions on W,

$$\text{i.e.}\quad \left.\begin{aligned} &\tilde{\nabla}^2\mathbf{U}-\tilde{\nabla}P=\mathbf{0},\quad \tilde{\nabla}\,.\,\mathbf{U}=0,\\ \text{and}\quad &\mathbf{U}=\mathbf{U}_w\quad \text{on } W,\end{aligned}\right\} \tag{5.1}$$

where $\mathbf{U}_w$ is the velocity of the walls W, which we have for the moment assumed to be both solid and rigid.

As in §3, we denote by $\mathfrak{F}$ the six-dimensional force-couple vector, i.e.

$$\mathfrak{F}=\begin{pmatrix}\mathbf{F}\\ \mathbf{G}\end{pmatrix}. \tag{5.2}$$

However, $\mathfrak{V}$ now denotes the six-dimensional velocity–angular velocity vector of the particle *relative* to the fluid. Thus we define

$$\mathfrak{V}=\begin{pmatrix}\mathbf{V}-\mathbf{U}_0\\ \mathbf{\Omega}\end{pmatrix}, \tag{5.3}$$

where $\mathbf{U}_0$ is the value of the velocity field $\mathbf{U}$ at the origin (and $\mathbf{V}$ the value of the particle velocity at the origin).

By using methods analogous to that used in §3 it may be shown that $\mathfrak{F}$ is given as an expansion in κ as

$$\begin{aligned}\mathfrak{F} = {} & -\mathfrak{K}\,.\,\mathfrak{V}+[\kappa\mathbf{p}_2:(\tilde{\nabla}\,\mathbf{U})_0+\kappa^2\mathbf{p}_3[3](\tfrac{1}{2}\tilde{\nabla}\tilde{\nabla}\,\mathbf{U})_0+\ldots]\\ & +\Sigma\,\kappa^{a+b+\ldots+n}\{\mathbf{p}_{a+1}[a+1]({}_{a+1}\mathbf{L}_b)[b]({}_b\mathbf{q}_{c+1})[c+1]\ldots({}_l\mathbf{q}_{m+1})[m+1]({}_{m+1}\mathbf{L}_n)[n]\}\\ & \times\{\mathbf{l}_n\,.\,\mathfrak{V}+\kappa({}_n\mathbf{q}_2):(\tilde{\nabla}\,\mathbf{U})_0+\kappa^2({}_n\mathbf{q}_3)[3](\tfrac{1}{2}\tilde{\nabla}\tilde{\nabla}\,\mathbf{U})_0+\ldots\},\end{aligned} \tag{5.4}$$

where the summation is taken over all integers $a, b \ldots n$ such that

$$a, c\ldots m \geqslant 0,\quad b, d\ldots l, n \geqslant 1. \tag{5.5}$$

The quantities $(\tilde{\nabla}\,\mathbf{U})_0$, $(\frac{1}{2}\tilde{\nabla}\tilde{\nabla}\,\mathbf{U})_0, \ldots$, are the values of derivatives of U at the origin.

Like the equation (3.32), the above equation (5.4) remains valid if some of the boundaries W are free surfaces so long as the flow field $\mathbf{U}$ itself satisfies the free-surface boundary conditions on all such surfaces.

It should be noted that when the arbitrary Stokes flow field is zero the above equation for $\mathfrak{F}$ reduces to the equation (3.32). Should there be no walls present (with $\mathbf{U} \neq \mathbf{0}$), the value of $\mathfrak{F}$ would be given by

$$\mathfrak{F} = -\mathfrak{K}\,.\,\mathfrak{V}+[\kappa\mathbf{p}_2:(\tilde{\nabla}\,\mathbf{U})_0+\kappa^2\mathbf{p}_3[3](\tfrac{1}{2}\tilde{\nabla}\tilde{\nabla}\,\mathbf{U})_0+\ldots]. \tag{5.6}$$

This result for the force and couple acting upon a particle in an arbitrary flow field has been given by Brenner (1964*b*) in a slightly different form.

As an example of a body placed in a Stokes flow with solid walls present, we consider a small body axisymmetric about the axis 1 and possessing fore–aft symmetry placed at rest between two parallel plates each perpendicular to the axis 3. Relative to the origin of co-ordinates at the centre of the body the positions of the two plates are defined by

$$\tilde{r}_3 = -d_1 \quad \text{and} \quad \tilde{r}_3 = +d_2 \quad (d_1 \quad \text{and} \quad d_2 > 0)$$

in terms of the $\tilde{r}$ variables. d_1 and d_2 are assumed to be of order unity (i.e. independent of the parameter κ). We also suppose that initially there exists an undisturbed Stokes velocity field $\mathbf{U}$ given by

$$\mathbf{U} = (U, 0, 0),$$

where $$U = \alpha + \beta \tilde{r}_3 + \gamma \tilde{r}_3^2,$$

α, β and γ being constants. The upper and lower plates are taken to have velocities $(\alpha + \beta d_2 + \gamma d_2^2)$ and $(\alpha - \beta d_1 + \gamma d_1^2)$ respectively so that the no-slip boundary conditions are satisfied (see figure 4). We now make use of the formula (5.4) to find

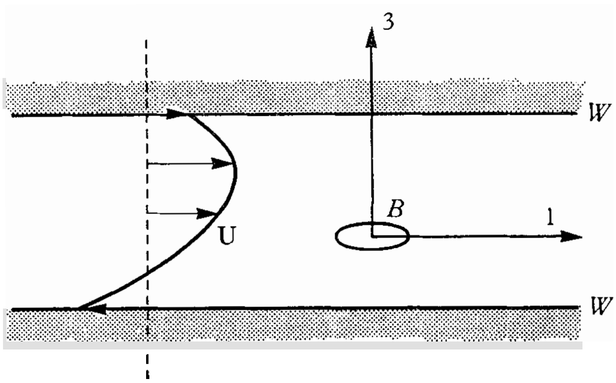

FIGURE 4

an expansion in κ for the dimensionless force $\mathbf{F}$ and couple $\mathbf{G}$ acting on the body. As in §4, we now make use of the symmetry of the body and of the walls to show that $\mathbf{F}$ and $\mathbf{G}$ are given by

$$\mathbf{F} = (F_1, 0, 0), \quad \mathbf{G} = (0, G_2, 0), \tag{5.7}$$

where
$$\left.\begin{aligned} F_1 &= \frac{a\alpha + \kappa^2 b\gamma + \kappa^3 c\beta}{1 + d\kappa + e\kappa^3 + f\kappa^4} + O(\kappa^5), \\ G_2 &= \kappa g\beta + \frac{\kappa^2 h\alpha}{1 + d\kappa} + O(\kappa^4), \end{aligned}\right\} \tag{5.8}$$

in which $a, b, c, \ldots, h$ are constants, independent of α, β, γ and κ. Wakiya (1957) has considered the present problem for the special case of the body being an axially symmetric spheroid. The values of F_1 and G_2 were respectively calculated to orders κ^4 and κ^3, when the spheroid was taken to be a disk. This was done for values of (α, β, γ) taken to be $(1, 0, 0)\,(1, 1, 0)$ and $(1, \frac{4}{3}, -\frac{1}{3})$. These results showed complete agreement with the equations (5.8). Wakiya computed the numerical coefficients occurring in his expansions for $d_2 = 3d_1$. For this case, one can make use of these expansions to find the values of the constants $a, b, c, \ldots, h$. Thus one obtains F_1 and G_2 as

$$\left.\begin{aligned} F_1 &= \frac{16(\alpha + 0{\cdot}3333\gamma\kappa^2 - 0{\cdot}1230\beta\kappa^3)}{(1 - 0{\cdot}5540\kappa + 0{\cdot}0272\kappa^3 - 0{\cdot}023\kappa^4)} + O(\kappa^5), \\ G_2 &= \frac{32}{3}\left\{\beta\kappa - \frac{0{\cdot}1846\alpha\kappa^2}{1 - 0{\cdot}5540\kappa}\right\} + O(\kappa^4). \end{aligned}\right\} \tag{5.9}$$

It may also be shown that for the special cases $d_1 = d_2$, for which the walls are centrally symmetric about the origin,

$$c = f = h = 0,$$

yielding

$$F_1 = \frac{a\alpha + \kappa^2 b\gamma}{1 + d\kappa + e\kappa^3} + O(\kappa^5),$$

$$G_2 = \kappa g\beta + O(\kappa^4). \tag{5.10}$$

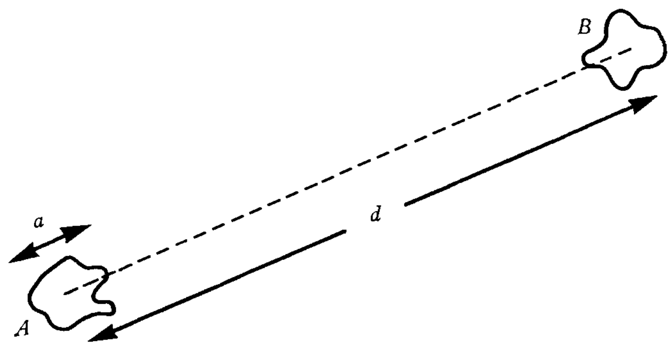

FIGURE 5

6. Two bodies in an arbitrary Stokes flow field

In this section we extend the theory developed in §3 to find the forces and couples acting upon each of two bodies of arbitrary shape placed in some arbitrary Stokes flow field there being no solid or other boundaries present (see figure 5). Two points, A and B, each instantaneously at rest, are taken inside each of the bodies. We let d be the distance between A and B, and a be the characteristic length dimension of the bodies which is assumed to be the same for each. The parameter κ, defined by the relation

$$\kappa = a/d, \tag{6.1}$$

is then assumed to be much smaller than unity. It is in terms of this parameter that we shall make the expansions.

$\mathbf{V}^A$ is defined to be the dimensionless velocity of the body A at the origin A and $\mathbf{\Omega}^A$ the angular velocity of the body A. The quantities $\mathbf{V}^B$ and $\mathbf{\Omega}^B$ are similarly defined with respect to the body B. As in §3, we define six-dimensional force-couple vectors as

$$\mathfrak{F}^A = \begin{pmatrix}\mathbf{F}^A \\ \mathbf{G}^A\end{pmatrix} \quad \text{and} \quad \mathfrak{F}^B = \begin{pmatrix}\mathbf{F}^B \\ \mathbf{G}^B\end{pmatrix}, \tag{6.2}$$

where $\mathbf{F}^A$, $\mathbf{G}^A$, $\mathbf{F}^B$, $\mathbf{G}^B$ are the forces and couples acting on the bodies A and B (about the respective origins).

The undisturbed dimensionless Stokes flow field $(\mathbf{U}, P)$ must satisfy

$$\nabla^2\mathbf{U} - \nabla P = \mathbf{0}, \quad \nabla . \mathbf{U} = 0 \tag{6.3}$$

everywhere whilst the complete dimensionless velocity field $(\mathbf{u}, p)$ satisfies

$$\left.\begin{aligned} &\nabla^2\mathbf{u} - \nabla p = \mathbf{0}, \quad \nabla . \mathbf{u} = 0, \\ &\mathbf{u} = \mathbf{V}^A + \mathbf{\Omega}^A \wedge \mathbf{r}^A \quad \text{on the surface of body } A, \\ &\mathbf{u} = \mathbf{V}^B + \mathbf{\Omega}^B \wedge \mathbf{r}^B \quad \text{on the surface of body } B, \\ &\mathbf{u} \sim \mathbf{U} \quad \text{as} \quad \mathbf{r}^A \to \infty \quad \text{and} \quad \mathbf{r}^B \to \infty, \end{aligned}\right\} \tag{6.4}$$

where $\mathbf{r}^A$ and $\mathbf{r}^B$ are dimensionless position vectors (made dimensionless by the length a) which are taken relative to the points A and B respectively.

We now define $(\mathbf{u}^*, p^*)$ to be the disturbance flow field so that

$$\mathbf{u} = \mathbf{U} + \mathbf{u}^*, \quad p = P + p^*. \tag{6.5}$$

This flow field must then satisfy Stokes equations with the boundary conditions

$$\left.\begin{aligned} &\mathbf{u}^* = -\mathbf{U} + \mathbf{V}^A + \mathbf{\Omega}^A \wedge \mathbf{r}^A \quad \text{on the surface of } A, \\ &\mathbf{u}^* = -\mathbf{U} + \mathbf{V}^B + \mathbf{\Omega}^B \wedge \mathbf{r}^B \quad \text{on the surface of } B, \\ &\mathbf{u}^* \to \mathbf{0} \quad \text{as} \quad \mathbf{r}^A \quad \text{and} \quad \mathbf{r}^B \to \infty. \end{aligned}\right\} \tag{6.6}$$

For simplicity and convenience we divide the flow field $(\mathbf{u}^*, p^*)$ into two parts $(\mathbf{u}_1^*, p_1^*)$ and $(\mathbf{u}_2^*, p_2^*)$ so that

$$\mathbf{u}^* = \mathbf{u}_1^* + \mathbf{u}_2^*, \quad p^* = p_1^* + p_2^*. \tag{6.7}$$

Each of these flow fields $(\mathbf{u}_1^*, p_1)$ and $(\mathbf{u}_2^*, p_2)$ are made to satisfy Stokes equations. The boundary conditions on $(\mathbf{u}_1^*, p_1^*)$ are taken to be

$$\left.\begin{aligned} &\mathbf{u}_1^* = -\mathbf{U} + \mathbf{V}^A + \mathbf{\Omega}^A \wedge \mathbf{r}^A \quad \text{on the surface of } A, \\ &\mathbf{u}_1^* = \mathbf{0} \quad \text{on the surface of } B, \\ &\mathbf{u}_1^* \to \mathbf{0} \quad \text{as} \quad \mathbf{r}^A \to \infty, \end{aligned}\right\} \tag{6.8}$$

which implies that the boundary conditions on $(\mathbf{u}_2^*, p_2^*)$ must be taken as

$$\left.\begin{aligned} &\mathbf{u}_2^* = \mathbf{0} \quad \text{on the surface of } A, \\ &\mathbf{u}_2^* = -\mathbf{U} + \mathbf{V}^B + \mathbf{\Omega}^B \wedge \mathbf{r}^B \quad \text{on the surface of } B, \\ &\mathbf{u}_2^* \to \mathbf{0} \quad \text{as} \quad \mathbf{r}^B \to \infty. \end{aligned}\right\} \tag{6.9}$$

$\mathfrak{F}_1^A$ and $\mathfrak{F}_1^B$ are taken to be those parts of $\mathfrak{F}^A$ and $\mathfrak{F}^B$ respectively which result from the flow field $(\mathbf{u}_1^*, p_1^*)$, whilst $\mathfrak{F}_2^A$ and $\mathfrak{F}_2^B$ are similarly defined as resulting from the flow field $(\mathbf{u}_2^*, p_2^*)$. Thus

$$\mathfrak{F}^A = \mathfrak{F}_1^A + \mathfrak{F}_2^A \quad \text{and} \quad \mathfrak{F}^B = \mathfrak{F}_1^B + \mathfrak{F}_2^B. \tag{6.10}$$

In order to evaluate $\mathfrak{F}_1^A$, $\mathfrak{F}_1^B$ and $\mathfrak{F}_2^A$, $\mathfrak{F}_2^B$ as expansions in the parameter κ, we define three regions of expansion as follows (see figure 6).

Region I for the inner expansion at A. This is valid in the neighbourhood of A, the independent variables are taken as $\mathbf{r}^A$, the position vector of a point relative to A.

Region II for the inner expansion at B. This is valid in the neighbourhood of B, the independent variables are taken as $\mathbf{r}^B$, the position vector of a point relative to B.

Region III for the outer expansion. This is valid outside regions I and II, the independent variables $\tilde{\mathbf{r}}^A$ being defined by

$$\tilde{\mathbf{r}}^A = \kappa \mathbf{r}^A. \tag{6.11}$$

We denote the value of $\tilde{\mathbf{r}}^A$ at the point B by the unit vector $\mathbf{R}$. The present problem is solved by expanding $(\mathbf{u}_1^*, p_1^*)$ and then $(\mathbf{u}_2^*, p_2^*)$ in terms of the parameter κ in the above three regions. In region I we require the boundary condition

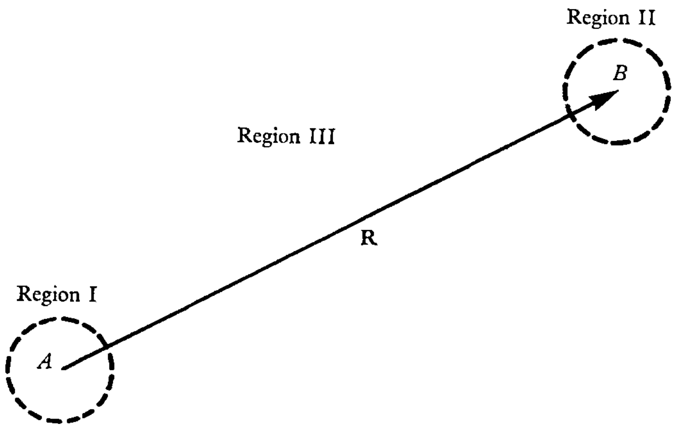

FIGURE 6

on the surface of body A to be satisfied whilst in region II the boundary condition on the surface of body B must be satisfied. The outer boundary conditions on the expansions in regions I and II are obtained by matching them onto the expansion in region III at the points A and B respectively. Thus proceeding as in § 3, it may be shown finally that $\mathfrak{F}_1^A$ is given by

$$\begin{aligned}\mathfrak{F}_1^A = & -\mathfrak{R}^A . \mathfrak{V}^A + [\kappa \mathbf{p}_2^A : (\tilde{\nabla} \mathbf{U})_A + \kappa^2 \mathbf{p}_3^A [3] (\tfrac{1}{2} \tilde{\nabla} \tilde{\nabla} \mathbf{U})_A + \ldots] \\ & + \Sigma\, \kappa^{a+b+\ldots+n} \{\mathbf{p}_{a+1}^A [a+1] ({}_{a+1}\mathbf{M}_b^B)[b]({}_b\mathbf{q}_{c+1}^B)[c+1] \ldots ({}_l\mathbf{q}_{m+1}^B)[m+1]({}_{m+1}\mathbf{M}_n^A)\} \\ & + [n]\{\mathbf{1}_n^A . \mathfrak{V}^A + [\kappa({}_n\mathbf{q}_2^A) : (\tilde{\nabla} \mathbf{U})_A + \kappa^2({}_n\mathbf{q}_3^A)[3](\tfrac{1}{2} \tilde{\nabla} \tilde{\nabla} \mathbf{U})_A + \ldots]\},\end{aligned} \tag{6.12}$$

whilst $\mathfrak{F}_2^A$ is given by

$$\begin{aligned}\mathfrak{F}_2^A = & \Sigma\, \kappa^{a+b+\ldots+t} \{\mathbf{p}_{a+1}^A [a+1]({}_{a+1}\mathbf{M}_b^B)[b]({}_b\mathbf{q}_{c+1}^B)[c+1]({}_{c+1}\mathbf{M}_d^A)[d] \ldots \\ & \ldots ({}_r\mathbf{q}_{s+1}^A)[s+1]({}_{s+1}\mathbf{M}_t^B)\}[t]\{\mathbf{1}_t^B . \mathfrak{V}^B + [\kappa({}_t\mathbf{q}_2^B) : (\tilde{\nabla} \mathbf{U})_B + \kappa^2({}_t\mathbf{q}_3^B)[3](\tfrac{1}{2} \tilde{\nabla} \tilde{\nabla} \mathbf{U})_B + \ldots]\},\end{aligned} \tag{6.13}$$

where
$$\mathfrak{V}^A = \begin{pmatrix} \mathbf{V}^A - \mathbf{U}_A \\ \boldsymbol{\Omega}^A \end{pmatrix} \quad \text{and} \quad \mathfrak{V}^B = \begin{pmatrix} \mathbf{V}^B - \mathbf{U}_B \\ \boldsymbol{\Omega}^B \end{pmatrix}, \tag{6.14}$$

the quantities $\mathbf{U}_A$, $(\tilde{\nabla} \mathbf{U})_A$, $(\tilde{\nabla} \tilde{\nabla} \mathbf{U})_A$, etc., being the values of $\mathbf{U}$ and its derivatives with respect to $\tilde{\mathbf{r}}^A$ evaluated at A, and the quantities $\mathbf{U}_B$, $(\tilde{\nabla} \mathbf{U})_B$, $(\tilde{\nabla} \tilde{\nabla} \mathbf{U})_B$, etc., the values of $\mathbf{U}$ and its derivatives evaluated at B. In the equation (6.12) the summation is taken over all integral values of $a, b, \ldots, n$ such that

$$a, c, \ldots, m \geqslant 0, \quad b, \ldots, l, n \geqslant 1, \tag{6.15}$$

whilst in equation (6.13), $a, c, \ldots, t$ take all integral values such that

$$a,\, c,\, \ldots,\, s \geqslant 0, \quad b,\, \ldots,\, r,\, t \geqslant 1. \tag{6.16}$$

The quantities $\mathfrak{K}^A$, $\mathbf{p}^A_n$, ${}_r\mathbf{q}^A_s$ and $\mathbf{l}^A_n$ are the quantities $\mathfrak{K}$, $\mathbf{p}_n$, ${}_r\mathbf{q}_s$ and $\mathbf{l}_n$ defined in §3 for the body A, whilst $\mathfrak{K}^B$, $\mathbf{p}^B_n$, ${}_r\mathbf{q}^B_s$ and $\mathbf{l}^B_n$ are similarly defined for the body B. All such quantities are independent of $\mathbf{R}$ and depend only on the shape of the body concerned. The tensors ${}_r\mathbf{M}^A_s$ and ${}_r\mathbf{M}^B_s$ occurring in equations (6.12) and (6.13) are functions only of $\mathbf{R}$ and are independent of the shapes of the bodies. It may be shown that

$$({}_1M^A_1)_{pi} = \left(\frac{\delta_{pi}}{R}+\frac{R_pR_i}{R^3}\right); \quad ({}_1M^A_2)_{pki} = \frac{\partial}{\partial R_k}\left(\frac{\delta_{pi}}{R}+\frac{R_pR_i}{R^3}\right);$$

$$({}_2M^A_1)_{qpi} = \frac{(-1)}{1!}\frac{\partial}{\partial R_q}\left(\frac{\delta_{pi}}{R}+\frac{R_pR_i}{R^3}\right); \quad ({}_2M^A_2)_{qpki} = \frac{(-1)}{1!}\frac{\partial^2}{\partial R_q \partial R_k}\left(\frac{\delta_{pi}}{R}+\frac{R_pR_i}{R^3}\right),$$

or in general that in polyadic notation

$${}_r\mathbf{M}^A_s = \frac{(-1)^{r-1}}{(r-1)!}\nabla^{r-1}{}_\circ\nabla^{s-1}{}_\circ\left[\frac{\mathbf{I}}{R}+\frac{\mathbf{RR}}{R^3}\right], \tag{6.17}$$

where ∇^{r-1} is defined by

$$\nabla^{r-1} = (\nabla\nabla\nabla\ldots\nabla), \tag{6.18}$$

the product of $(r-1)$ factors and the symbol ${}_\circ$ denotes the transposing of an index without any other operation as in the following expression

$$(\nabla_\circ \nabla_\circ \mathbf{A})_{ijklm} = \frac{\partial^2}{\partial r_i \partial r_j}\frac{\partial}{\partial r_l}A_{km}. \tag{6.19}$$

The quantity ${}_r\mathbf{M}^B_s$ is an $(r+s)$-order tensor defined in the same manner as ${}_r\mathbf{M}^A_s$ but with $\mathbf{R}$ replaced by $-\mathbf{R}$. Thus it is observed that

$${}_r\mathbf{M}^B_s = (-1)^{r+s}\,{}_r\mathbf{M}^A_s. \tag{6.20}$$

Adding the equations (6.12) and (6.13) the value of $\mathfrak{F}^A$ is obtained as

$$\begin{aligned}\mathfrak{F}^A = {} & -\mathfrak{K}^A.\mathfrak{V}^A+[\kappa\mathbf{p}^A_2:(\tilde{\nabla}\mathbf{U})_A+\kappa^2\mathbf{p}_3[3](\tfrac{1}{2}\tilde{\nabla}\tilde{\nabla}\mathbf{U})_A+\ldots]\\ & +\Sigma\,\kappa^{a+b+\ldots+n}\{\mathbf{p}^A_{a+1}[a+1]({}_{a+1}\mathbf{M}^B_b)[b]({}_b\mathbf{q}^B_{c+1})[c+1]({}_{c+1}\mathbf{M}^A_d)[d]({}_d\mathbf{q}^A_{e+1})[e+1]\ldots\\ & \quad({}_l\mathbf{q}^B_{m+1})[m+1]({}_{m+1}\mathbf{M}^A_n)\}[n]\{\mathbf{l}^A_n.\mathfrak{V}^A+[\kappa({}_n\mathbf{q}^A_2):(\tilde{\nabla}\mathbf{U})_A\\ & \qquad\qquad +\kappa^2({}_n\mathbf{q}^A_3)[3](\tfrac{1}{2}\tilde{\nabla}\tilde{\nabla}\mathbf{U})_A+\ldots]\}\\ & +\Sigma\,\kappa^{\bar{a}+\bar{b}+\ldots+\bar{l}}\{\mathbf{p}^A_{\bar{a}+1}[\bar{a}+1]({}_{\bar{a}+1}\mathbf{M}^B_{\bar{b}})[\bar{b}]({}_{\bar{b}}\mathbf{q}^B_{\bar{c}+1})[\bar{c}+1]({}_{\bar{c}+1}\mathbf{M}^A_{\bar{d}})[\bar{d}]\ldots\\ & \quad({}_{\bar{r}}q^A_{\bar{s}+1})[\bar{s}+1]({}_{\bar{s}+1}\mathbf{M}^B_{\bar{t}})\}[\bar{t}]\{\mathbf{l}^B_{\bar{t}}.\mathfrak{V}^B+[\kappa({}_{\bar{t}}\mathbf{q}^B_2):(\tilde{\nabla}\mathbf{U})_B+\kappa^2({}_{\bar{t}}\mathbf{q}^B_3)[3](\tfrac{1}{2}\tilde{\nabla}\tilde{\nabla}\mathbf{U})_B+\ldots]\},\end{aligned} \tag{6.21}$$

where

$$\left.\begin{aligned} & a,\, c,\, \ldots,\, m \geqslant 0, \quad b,\, \ldots,\, l,\, n \geqslant 1;\\ & \bar{a},\, \bar{c},\, \ldots,\, \bar{s} \geqslant 0, \quad \bar{b},\, \ldots,\, \bar{r},\, \bar{t} \geqslant 1.\end{aligned}\right\} \tag{6.22}$$

It may be shown that the formula for $\mathfrak{F}^B$ is obtained by interchanging the A and B in equation (6.21).

7. Examples of two bodies at large separation

In this section we make use of results obtained in §6 to find the force and couple acting upon each of two axially symmetric bodies with fore–aft symmetry which are translating without rotation in a fluid in such a manner that the distance between them is large compared with their size. For simplicity we assume that the two bodies are of identical shape and are translating with identical velocities in the direction of the line joining their centres. The axes of

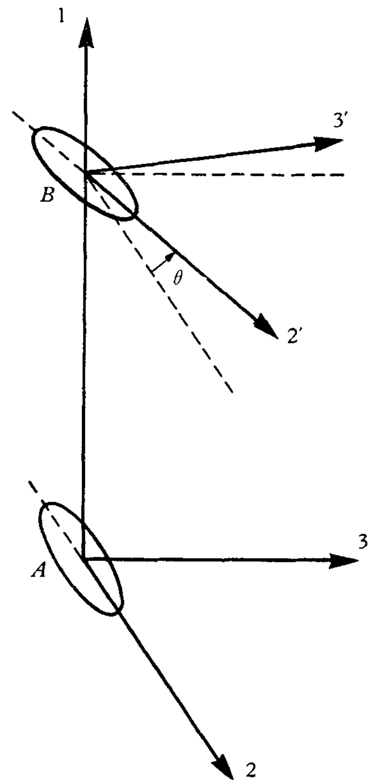

FIGURE 7.

symmetry of each of the bodies are assumed to be perpendicular to their velocities (figure 7). At the centres A and B of the bodies we take axes (1, 2, 3) and (1′, 2′, 3′) respectively such that 1 and 1′ lie along AB and the axes 2 and 2′ are symmetry axes for the bodies. We let θ be the angle between axes 2 and 2′.

Since both bodies A and B are centrally symmetric

$$ {}_r\mathbf{q}_s^A = {}_r\mathbf{q}_s^B = 0 \tag{7.1} $$

if $(r+s)$ is odd. Since the velocity $\mathbf{V}$ has components $(V, 0, 0)$ relative to the $(1, 2, 3)$ set of axes, the values of the force $\mathbf{F}^A$ and couple $\mathbf{G}^A$ acting upon the body A may, by equations (6.12), (6.20) and (7.1), be written in the forms

$$ \mathbf{F}^A = (F^A, 0, 0), \quad \mathbf{G}^A = (G^A, 0, 0), \tag{7.2} $$

where

$$
\begin{aligned}
F^A = 8\pi[&\{({}_1q_1^A)_{11}\}V + \kappa\{({}_1q_1^A)_{11}({}_1M_1^A)_{11}({}_1q_1^B)_{11}\}V + \kappa^2\{({}_1q_1^A)_{11}({}_1M_1^A)_{11}({}_1q_1^B)_{11} \\
&\quad \times ({}_1M_1^A)_{11}({}_1q_1^A)_{11}\}V + \kappa^3\{({}_1q_1^A)_{11}({}_1M_1^A)_{11}({}_1q_1^B)_{11}({}_1M_1^A)_{11}({}_1q_1^A)_{11}({}_1M_1^A)_{11}({}_1q_1^B)_{11} \\
&+ ({}_1q_3^A)_{1pqr}({}_3M_1^A)_{rqp1}({}_1q_1^B)_{11} + ({}_1q_1^A)_{11}({}_1M_3^A)_{1pqr}({}_3q_1^B)_{rqp1}\}V \\
&+ \kappa^4\{({}_1q_1^A)_{11}({}_1M_1^A)_{11}({}_1q_1^B)_{11}({}_1M_1^A)_{11}({}_1q_1^A)_{11}({}_1M_1^A)_{11}({}_1q_1^B)_{11}({}_1M_1^A)_{11}({}_1q_1^A)_{11} \\
&+ ({}_1q_3^A)_{1pqr}({}_3M_1^A)_{rqp1}({}_1q_1^B)_{11}({}_1M_1^A)_{11}({}_1q_1^A)_{11} \\
&+ ({}_1q_1^A)_{11}({}_1M_3^A)_{1pqr}({}_3q_1^B)_{rqp1}({}_1M_1^A)_{11}({}_1q_1^A)_{11} + ({}_1q_1^A)_{11}({}_1M_1^A)_{11}({}_1q_1^B)_{11}({}_1M_3^A)_{1pqr} \\
&\quad \times ({}_3q_1^A)_{rqp1} + ({}_1q_1^A)_{11}({}_1M_1^A)_{11}({}_1q_3^B)_{1pqr}({}_3M_1)_{rqp1}({}_1q_1^A)_{11} \\
&- ({}_1q_1^A)_{11}({}_1M_2^A)_{1pq}({}_2q_2^B)_{qpsr}({}_2M_1^A)_{rs1}({}_1q_1^A)_{11}\}V] + O(\kappa^5)
\end{aligned}
\tag{7.3}
$$

and

$$
\begin{aligned}
G^A = 8\pi[&\kappa^2\{(\boldsymbol{\epsilon} : {}_2\mathbf{q}_2^A)_{1pq}({}_2M_1^A)_{qp1}({}_1q_1^B)_{11}\}V + \kappa^3\{(\boldsymbol{\epsilon} : {}_2\mathbf{q}_2^A)_{1pq}({}_2M_1^A)_{qp1} \\
&\times ({}_1q_1^B)_{11}({}_1M_1^A)_{11}({}_1q_1^A)_{11}\}V + \kappa^4\{(\boldsymbol{\epsilon} : {}_2\mathbf{q}_2^A)_{1pq}({}_2M_1^A)_{qp1}({}_1q_1^B)_{11}({}_1M_1^A)_{11}({}_1q_1^A)_{11} \\
&\times ({}_1M_1^A)_{11}({}_1q_1^B)_{11} + (\boldsymbol{\epsilon} : {}_2\mathbf{q}_4^A)_{1pqrs}({}_4M_1^A)_{srqp1}({}_1q_1^B)_{11} \\
&+ (\boldsymbol{\epsilon} : {}_2\mathbf{q}_2^A)_{1pq}({}_2M_3^A)_{qptsr}({}_3q_1^B)_{rst1}\}V] + O(\kappa^5),
\end{aligned}
\tag{7.4}
$$

all the tensors being expressed relative to the (1, 2, 3) set of axes. Since the two bodies are of identical shape

$$({}_1q_1^A)_{11} = ({}_1q_1^B)_{1'1'} = ({}_1q_1^B)_{11}. \tag{7.5}$$

Thus, by making use of this relation, it is seen that the equation (7.3) for F^A is of the form

$$F^A = \frac{V}{a + b\kappa + c\kappa^3 + d\kappa^4} + O(\kappa^5), \tag{7.6}$$

where a, b, c, d are constants. Each term occurring in (7.3) up to and including terms of order κ^4 may be shown to be independent on the angle θ; e.g. the term

$$T = -8\pi\kappa^4({}_1q_1^A)_{11}({}_1M_2^A)_{1pq}({}_2q_2^B)_{qpsr}({}_2M_1^A)_{rs1}({}_1q_1^A)_{11}V$$

may be written in terms of the (1′, 2′, 3′) set of axes as

$$T = -8\pi\kappa^4({}_1q_1^A)_{1'1'}({}_1M_2^A)_{1'p'q'}({}_2q_2^B)_{q'p's'r'}({}_2M_1^A)_{r's'1'}({}_1q_1^A)_{1'1'}V,$$

in which we see that each factor is independent of θ since $({}_1q_1^A)_{1'1'} = ({}_1q_1^A)_{11}$. Hence the constants a, b, c and d appearing in equation (7.6) are functions only of the shape of the bodies A and B and are independent of the angle θ. This result is in agreement with the calculations performed by Wakiya (1965) for a pair of spheroids.

By making further use of the symmetry of the bodies, the equation (7.4) for G^A reduces to

$$G^A = 8\pi\kappa^4(\boldsymbol{\epsilon} : {}_2\mathbf{q}_2^A)_{1pq}({}_2M_3^A)_{qptsr}({}_3q_1^B)_{rst1}V + O(\kappa^5). \tag{7.7}$$

For the body A, the tensor $(\boldsymbol{\epsilon} : {}_2\mathbf{q}_2^A)_{1pq}$ [relative to the (1, 2, 3) set of axes] is zero unless (p, q) is equal to (2, 3) or (3, 2). However, the tensor $({}_2M_3^A)_{qptsr}$ is also zero unless the set of indices $(qptsr)$ contains an even number of 2's and of 3's. Thus the contribution to G^A is zero unless (tsr) is a permutation of (1 2 3). Expressing the tensor $({}_3q_1^B)$ relative to the $(1', 2', 3')$ axes we have

$$({}_3q_1^B)_{rst1} = \alpha_{ra'}\alpha_{sb'}\alpha_{tc'}\alpha_{1d'}({}_3q_1^B)_{a'b'c'd'}, \tag{7.8}$$

where α_{ij} is the transformation matrix from one set of axes to the other, i.e.

$$\alpha_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}. \tag{7.9}$$

Hence

$$({}_3q_1^B)_{rst1} = \alpha_{ra'}\alpha_{sb'}\alpha_{tc'}({}_3q_1^B)_{a'b'c'1'}. \tag{7.10}$$

The symmetry of the body B relative to the $(1', 2', 3')$ set of axes requires that $({}_3q_1^B)_{a'b'c'1'}$ is zero unless the set (a', b', c') contains an even number of 2's and 3's and an odd number of 1's. Thus since we have also shown that $({}_3q_{\text{II}}^B)_{rst1}$ is zero unless (tsr) is a permutation of (1 2 3) it follows that each non-zero component of $({}_3q_1^B)_{rst1}$ must be proportional to $\sin\theta\cos\theta$ (see equation (7.9)). Thus the equation (7.7) for G^A is of the form

$$G^A = \kappa^4 f \sin\theta\cos\theta V + O(\kappa^5), \tag{7.11}$$

where f is a constant independent of θ.

There is a discrepancy between the equation (7.11) and the corresponding result for two spheroids obtained by Wakiya (1965) who obtained the dependence upon θ as being proportional to $\sin\theta\cos\theta(6\cos^2\theta - 1)$.†

8. Discussion

In this paper we have found the force and couple acting upon a solid body of arbitrary shape which moves in a fluid which is itself undergoing some arbitrary motion, there also being solid and free surface boundaries present. Such a solution was obtained as an expansion in a small parameter κ defined as the ratio of particle size to the length scale of the undisturbed fluid motion. This expansion contained an infinite number of coefficients each of which could be obtained by solving a well-defined Stokes flow problem relating *either* to the body *or* to the surrounding boundaries. Thus in order to obtain a solution to any particular order in κ, we have reduced our rather complicated problem to one of solving a set of simpler problems.

Even when it is not possible to obtain any of the coefficients occurring in the solution, such results may, as we have seen, be used to find the form of the expansion, thus showing for instance the order of magnitude of the wall effects. With such a knowledge of the form of the expansion, it would then be possible to find values for the unknown coefficients by experimental means. Such a procedure would be particularly useful for cases in which the body and the surrounding walls possessed some form of symmetry.

As has been shown in §§4, 5 and 7, the theory may also be used as a partial check upon the correctness of the solution for the many problems for which the

† It would appear that the values of the constants b_{021}, b_{003}, c_{012} and c_{030} given by Wakiya (see equation (4.5) on page 1511 of this reference) are incorrect.

actual coefficients in the expansion in κ have been obtained by the method of reflexion and other means.

Because of the very general nature of the results obtained in this paper, they may be used in further theoretical investigations on the motion of particles of arbitrary shape. Thus in a forthcoming paper by Cox & Brenner, these results will be used to investigate the effect of fluid inertia upon the motion of a particle moving in the neighbourhood of walls, with special application to the lateral migration of solid particles in Poiseuille flows.

This work was supported by the National Science Foundation under Grant no. NSF:GK-56.

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